Lect22-0407 Connect

Thursday, April 7, 2016 3:24 PM

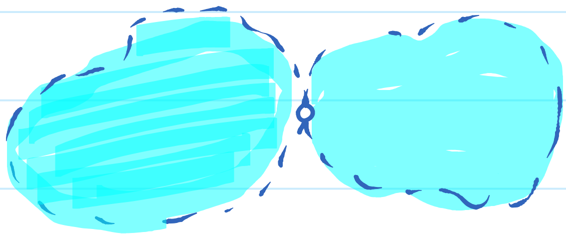
Recall that

X is disconnected $\Leftrightarrow$ $\exists\ \phi \neq U, V \subsetneqq X$ such that U, V are both open and closed.

It is convenient to use the negation below

Definition (useful in doing proof)
$(X, \mathcal{T})$ is connected if $\forall\ U \subset X$ that is both open and closed in X, $U = \phi$ or $U = X$.
Note that no need to mention V in above

Qu. Draw a picture of a disconnected subset in $\mathbb{R}^2$, which is "almost connected"



From this example, if $X = A \cup B$ with $A \cap B = \phi$ the condition on A, B will determine whether X is connected or disconnected.

Touching boundaries

Thursday, April 7, 2016 3:24 PM

Theorem X be connected $\iff \forall\ A, B \neq \phi$ if $X = A \cup B$ with $A \cap B = \phi$ then $\bar{A} \cap B \neq \phi$ or $A \cap \bar{B} \neq \phi$

Example

① $X = (0,2) = \underbrace{(0,1)}_{A} \cup \underbrace{[1,2)}_{B}$ connected

$\bar{A} = (0,1]$, $B = [1,2)$, $\bar{A} \cap B = \{1\}$

② $X = \underbrace{(0,1)}_{A} \cup \underbrace{(1,2)}_{B}$ disconnected

$\bar{A} = (0,1)$, $B = (1,2)$ $\bar{A} \cap B = \phi$

Main Idea

By $X = A \cup B$ and $A \cap B = \phi$,

$A = X \setminus B$

If $A \cap \bar{B} = \phi$ then $A \subset X \setminus \bar{B}$

$X \setminus B \subset X \setminus \bar{B}$, $\therefore B \supset \bar{B}$

Thus, B is closed and A is open

Therefore $\bar{A} \cap B = \phi$ and $A \cap \bar{B} = \phi$

implies both A, B are open (and closed)

The above argument clearly goes backward.

Properties

Thursday, April 7, 2016 3:25 PM

Qu. What is the Intermediate Value Theorem? And it analogue in high dimension?

Theorem If X is connected and $f: X \to Y$ is continuous then $f(X)$ is connected.

Proof Let $S \subset f(X)$ is both open and closed in Y

i.e. $S = G \cap f(X)$ $G \in \mathcal{J}_Y$

$= F \cap f(X)$ $X \setminus F \in \mathcal{J}_Y$

$f^{-1}(S) = f^{-1}(G) = f^{-1}(F)$

both open & closed in X

$\therefore f^{-1}(S) = \phi$ or $f^{-1}(S) = X$

i.e. Both $G, F \subset Y \setminus f(X)$ $S = \phi$

or both $G, F \supset f(X)$ $S = f(X)$

Theorem X is disconnected $\Longleftrightarrow$

$\exists$ surjective continuous $f: X \longrightarrow (\{-1,1\}, \text{discrete})$

"$\Rightarrow$" Let $\phi \neq U \subsetneq X$ be both open & closed

Then define $f(x) = \begin{cases} -1 & x \in U \\ 1 & x \notin U \end{cases}$ will do.

"$\Leftarrow$" Simply take $U = f^{-1}(-1)$, $V = f^{-1}(1)$, $U \cup V = X$

$U, V \neq \phi$ because f is surjective

They are open and closed because $\{-1\}, \{1\}$ are both open and closed in discrete topology.

Connected Component

Thursday, April 7, 2016 8:35 PM

Connected component of $x_0 \in X$

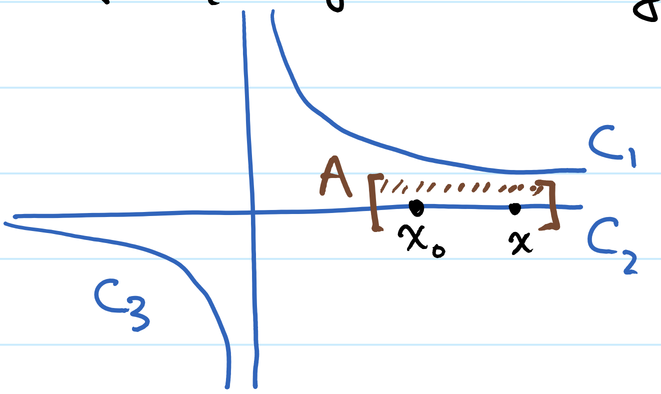
① C is the maximal / largest connected subset of X containing x_0

② $C = \bigcup \{\text{connected subsets containing } x_0\}$

③ Define $\sim$ on X by $x \sim y$ if $\exists$ connected $A \subset X$ such that $x, y \in A$

Then $C = [x_0]$

Example. $X = \{(x,y) \in \mathbb{R}^2 : xy = 0 \text{ or } xy = 1\}$



Intuitive picture

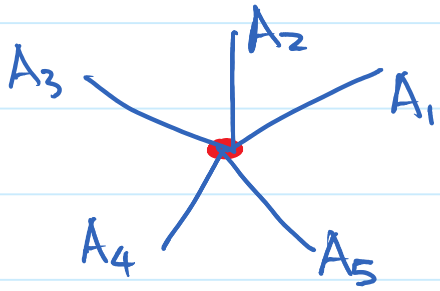
Qu. How do we know that C_1, C_2, C_3 are connected.

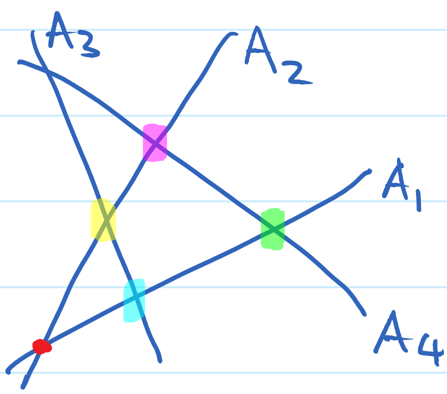
Qu. What must we do when we have 3 def's ?

Theorem Let $A_\alpha \subset X$ be connected subsets with

(i) $\bigcap_\alpha A_\alpha \neq \phi$ or (ii) $A_\alpha \cap A_\beta \neq \phi$ $\forall$ pair α, β

Then $\bigcup_\alpha A_\alpha$ is connected





Def equiv

Thursday, April 7, 2016 9:35 PM

① ⇒ ② By condition (i),

$$\bigcup\{A \subset X : x_0 \in A \text{ and } A \text{ is connected}\}$$

is a connected set containing x_0.

② ⇒ ③ First, we need condition (ii)

to show that $x \sim y$ and $y \sim z \Rightarrow x \sim z$

Then, we show

$$[x_0] = \bigcup\{A \subset X : x_0 \in A \text{ and } A \text{ is connected}\}$$

"$\supset$" If $x \in$ RHS then $\exists\, A \subset X$ with

$x_0 \in A$ and A is connected at $x \in A$

By definition of $\sim$, $x \sim x_0$, $\therefore x \in [x_0]$

"$\subset$" Similar

③ ⇒ ① By def of $\sim$ and maximality of C

$x \in [x_0] \Rightarrow x \in A \subset C$

$\therefore [x_0] \subset C$

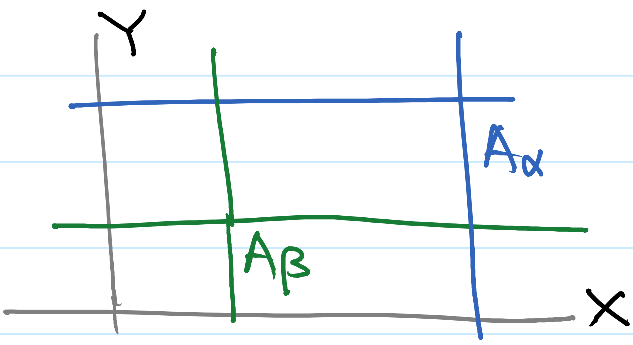
On the other hand, $x \in C \Rightarrow x \sim x_0$

$C \subset [x_0]$

Another use If X, Y are connected then

$X \times Y$ is connected

The idea is given by the picture



Proof

Thursday, April 7, 2016   8:35 PM

Note: (i) is a particular case of (ii),

Sufficient to show (ii) $\Rightarrow A = \bigcup_\alpha A_\alpha$ is connected

Let $S \subset A$ be both open and closed in A

$\therefore$ $S \cap A_\alpha$ is both open and closed $\forall \alpha$

$\therefore$ $\forall \alpha \in I$, $S \cap A_\alpha = \phi$ or $S \cap A_\alpha = A_\alpha$

$$\Downarrow ?$$

$$\underbrace{\forall \alpha \in I \quad S \cap A_\alpha = \phi} \quad \text{or} \quad \underbrace{\forall \alpha \in I \quad S \cap A_\alpha = A_\alpha}$$

$$S = S \cap A = S \cap \left(\bigcup_{\alpha \in I} A_\alpha\right) = \bigcup_{\alpha \in I} S \cap A_\alpha = \phi$$

or

$$S = \cdots = \bigcup_{\alpha \in I} S \cap A_\alpha = \bigcup_{\alpha \in I} A_\alpha = A$$

In ? above, logical, it may happen

$S \cap A_\alpha = \phi$ for some α; while $S \cap A_\alpha = A_\alpha$ for other α.

We need the assumption $A_\alpha \cap A_\beta \neq \phi \ \forall$ pair α, β

Suppose $\exists$ particular α with $S \cap A_\alpha = \phi$

Then $S \cap (A_\alpha \cap A_\beta) \subset A_\alpha \cap A_\beta = \phi$

$$\Vert \quad (S \cap A_\beta) \cap A_\alpha = \begin{cases} A_\beta \cap A_\alpha \neq \phi \\ \phi \cap A_\alpha = \phi \end{cases}$$

Thus $\forall$ arbitrary $\beta \in I$, $S \cap A_\beta = \phi$

Using the contrapositive, we get

$\exists \beta$ with $S \cap A_\beta = A_\beta \Rightarrow \forall \alpha$, $S \cap A_\alpha = A_\alpha$